Ex. Laplace with weak anchoring

Let $\Omega = [0,1] \times [0,1]$ and $u : \Omega \to \mathbb{R}$ be such that $u \in H^1(\Omega)$, and let an energy be given by

$$E[u] := \int_\Omega |\nabla u|^2 + \int_\Gamma |u - u_0|^2 - \int_\Omega fu - \int_\Gamma gu, \tag{1}$$

where $\Gamma = \partial\Omega$ and $f, g, u_0 \in H^1(\Omega)$ are fixed. We can take the variational derivative of this and set it equal to zero, thus obtaining the weak formulation:

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Gamma (u - u_0)v = \int_\Omega fv + \int_\Gamma gv, \qquad \forall v \in H^1(\Omega). \tag{2}$$

To obtain the strong formulation, we first need to integrate by parts using the Divergence Theorem. When we do this, the equation becomes

$$\int_\Omega (-\nabla^2 u)v + \int_\Gamma (\nabla u \cdot \nu)v + \int_\Gamma (u - u_0)v = \int_\Omega fv + \int_\Gamma gv, \qquad \forall v \in H^1(\Omega), \tag{3}$$

where ν is the outward-pointing unit vector normal to Γ. Now, set $(u - u_0) + \nabla u \cdot \nu = g$ on Γ, and we can substitute again to obtain

$$\int_\Omega (-\nabla^2 u)v = \int_\Omega fv, \qquad \forall v \in H^1(\Omega), \tag{4}$$

Using the usual line of argumentation, this can be equivalently written as

$$\begin{aligned} -\nabla^2 u &= f, \qquad \text{in } \Omega, \\ u + \nabla u \cdot \nu &= g + u_0, \qquad \text{on } \Gamma. \end{aligned} \tag{5}$$

Now, let us simulate this in Firedrake. Let $u_0 := (1,1)^T \nu$, and let $u_m(x,y) := \sin(x) + \cos(y)$ be the manufactured solution. Then

$$f = -\nabla^2 u_m = -(-\sin(x) - \cos(y)) = \sin(x) + \cos(y) = u_m, \tag{6}$$

and

$$g = u_m + \nabla u_m \cdot \nu - u_0 = u_m + (\cos(x), -\sin(y))^T \nu - (1,1)^T \nu. \tag{7}$$